

Differential Geometry of Finsler Space with Special Metrics

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1 Abstract

This study aims to investigate the theoretical framework of Finsler metrics with the following objectives.

1. **Chapter 1** presents the theoretical and historical foundation of the thesis, tracing the development of Finsler geometry as a generalization of Riemannian geometry. It covers the transition from isotropic to anisotropic spaces, the Finsler metric function $F(x, y)$, fundamental definitions, and key Finsler metric types, along with a literature survey, applications, and future research directions.
2. **Chapter 2** investigates projective changes between specific (α, β) -metrics. We establish necessary and sufficient conditions for projective transformations between a special cubic metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and a Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$, and similarly between the square root metric $F = \sqrt{\alpha(\alpha + \beta)}$ and a Kropina metric. Foundational conditions for projective relatedness are derived, along with requirements for isotropic S -curvature and isotropic Berwald curvature.
3. **Chapter 3** introduces a new Finsler metric, the exponential (α, β) -metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$, and examines its geometric structure. We derive necessary and sufficient conditions for this metric to define a Berwald space and a Douglas space. Additionally, special cases are analyzed to determine conditions under which the metric reduces to a weakly Berwald space.
4. **Chapter 4** explores the curvature properties of two Finslerian structures: the cubic metric and the generalized Matsumoto metric. We investigate locally projectively flat structures under constant Ricci curvature, and determine conditions under which these metrics admit vanishing flag curvature ($K = 0$).
5. **Chapter 5** investigates locally dually flat and projectively flat properties of cubic and exponential (α, β) -metrics under isotropic S -curvature. We derive necessary and sufficient conditions for these metrics to exhibit dual flatness and provide mathematical proofs for conditions under which they reduce to locally Minkowskian spaces. These results contribute to the classification of special (α, β) -metrics and their curvature invariants.
6. **Chapter 6** presents a broader class of Finsler metrics extending beyond the traditional (α, β) -framework, constructed using a Riemannian metric alongside two differential 1-forms, $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$. We derive the spray coefficients and Cartan connection to elucidate the internal geometry of these structures. Necessary and sufficient conditions for projective correspondences are established, yielding several novel Douglas metrics. This work extends Finsler geometry into multiformal manifolds, offering new tools for the study of geodesic-preserving transformations.
7. **Chapter 7** investigates the S -curvature and Mean Berwald curvature for a class of homogeneous exponential (α, β) -metrics. Exploiting the fact that the curvature of homogeneous spaces is governed by the algebraic structure of the underlying Lie algebra, we derive an explicit analytical formula quantifying the rate of distortion change along geodesics. These results are further utilized to provide a complete characterization of the Mean Berwald curvature.
8. **Chapter 8** investigates the framework of $\mathbb{R}$ -complex Finsler spaces, a generalization of complex Finsler geometry with scalar constraints restricted to the real field

$\lambda \in \mathbb{R}$. We derive the fundamental metric tensors, their determinants, and inverses for a specialized cubic-type metric within this framework. Necessary criteria for these metrics to exhibit non-Hermitian characteristics are established, identifying where standard complex symmetries break down. Specific Riemannian examples of Hermitian cubic structures are provided, offering a complete analytical profile of the interplay between complex structures and cubic-type deformations.

9. **Chapter 9** introduces a new category of Finsler metrics arising from a Kropina-Randers type modification, constructed using a Riemannian metric $\rho = \sqrt{a_{ij}(x)y^i y^j}$ combined with two differential 1-forms, $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$. This construction generates a broader family generalizing the traditional (ρ, β) -metric framework. We analyze the geometric properties of this class and derive necessary and sufficient conditions for local projective flatness, along with explicit criteria for these metrics to be of the Douglas type. These results deepen the understanding of projective geometry in Finsler spaces.
10. The **final chapter** synthesizes the key findings of the thesis, summarizing the progress made in exploring the differential geometry of Finsler spaces characterized by novel metrics. It highlights the significance of the newly introduced metric classes and their geometric properties, and reflects on the implications of the projective transformations, curvature conditions, and metric classifications established throughout this work. The chapter concludes by identifying promising directions for future research, proposing potential extensions and applications that broaden the scope of Finsler geometry in both theoretical and practical domains.

2 State of the Art

In ancient times, the field of mathematics was primarily categorized into two core disciplines: geometry and the study of numbers. Geometry is a discipline of mathematics that studies the mathematical description, classification, and relationships between shapes. At its essence, geometry defines the relationships between two points. Conversely, differential geometry utilizes differential calculus and integral calculus as methods to explore geometric issues.

The conceptual origins of Finsler geometry trace back to Bernhard Riemann's landmark lecture of 1854, in which he laid the foundations of modern differential geometry and acknowledged the possibility of a more general metric structure beyond the quadratic Riemannian case. It was not until 1918 that Paul Finsler revived this generalized framework through his doctoral dissertation under the supervision of Carathéodory, thereby establishing the field now known as Finsler geometry. The term Finsler space was formally introduced by J. H. Taylor in 1928. In 1934, Élie Cartan strengthened the theoretical foundations by introducing an axiomatic system that uniquely determined a Finsler connection from the fundamental function F [Cartan 1934], and later, Shiing-Shen Chern, together with David Bao and Zhongmin Shen, established the modern framework of the subject, which remains the standard reference for researchers worldwide.

In recent years, the field has expanded significantly beyond pure mathematics, finding important applications in general relativity, Finsler gravity, gravitational wave theory, cosmology, and mathematical biology. Researchers today continue to explore open problems in the classification of Finsler metrics, comparison theorems, and the geometric analysis

on Finsler manifolds, confirming that Finsler geometry remains one of the most active and promising frontiers in modern differential geometry.

2.1 Finsler Space

Definition 1 A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ with the following properties:

1. F is smooth on TM_0 ,
2. F is positively 1-homogeneous on the fibers of the tangent bundle TM and
3. The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0 .

The pair $F^n = (M, F)$ is called a Finsler space of dimension n . F is called the fundamental function, and g_{ij} is called the fundamental tensor of the Finsler space F^n .

2.2 Special Finsler Metrics

1. Randers metric

In 1854, B. Riemann sought to explore metrics in their general forms and introduced the concept of curvature for a specific category of metrics known as Riemann metrics [5]. The Randers metric was first presented in 1941 by Norwegian physicist Randers [13] while he was engaged in research on general relativity. Randers metrics are essential and intrinsic Finsler metrics that are defined as the combination of a Riemann metric and a 1-form. The Randers metric exemplifies the most straightforward case of a non-Riemannian Finsler metric on a smooth manifold M , created using a Riemannian metric along with a 1-form defined by

$$F = \alpha + \beta,$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ is a 1-form such that $\|\beta\| < 1$.

2. Kropina metric

Kropina metrics were first introduced by L. Berwald in relation to a two-dimensional Finsler space featuring rectilinear extremals, and they were studied by V. K. Kropina [7]. In 1973, Hashiguchi and colleagues investigated two-dimensional Landsberg metrics and referred to them as the Kropina metrics. The Kropina metric is characterized as follows:

$$F = \frac{\alpha^2}{\beta}.$$

Subsequently, the Kropina metric is extended to a more general metric known as the generalized Kropina metric. A Finsler metric is referred to as a generalized m-Kropina metric when it is expressed in the form

$$F = \frac{\alpha^{m+1}}{\beta^m} \quad (m \neq 0, -1),$$

where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ is a 1-form metric.

3. Matsumoto-metric

Matsumoto established a precise model of a Finsler surface in 1989 and introduced a particular type of Finsler metric, as illustrated by:

$$F = \frac{\alpha^2}{\alpha - \beta}. \quad (1)$$

Furthermore, several researchers have investigated a generalized Matsumoto metric, defined as:

$$F = \frac{\alpha^{m+1}}{(\alpha - \beta)^m} \quad (2)$$

where m is an integer $\neq 0, 1, -1$. This generalized form extends the classical Matsumoto metric by introducing a parameter m , thereby enabling the consideration of a more expansive range of geometric structures and curvature behaviors. It facilitates the exploration of how the interaction between the Riemannian metric α and the 1-form β influences the properties of Finsler spaces.

4. (α, β) -metric

(α, β) -Finsler metrics represent a specific category of Finsler metrics that originate from a Riemannian metric α and a 1-form β . These metrics are suitable for characterizing anisotropic spaces where behaviors that depend on direction are important, owing to their added flexibility compared to Riemannian metrics. An (α, β) -metric on a connected smooth n -manifold M is a Finsler metric and can be represented as

$$F = \alpha \phi(s); \quad s = \frac{\beta}{\alpha}.$$

5. Cubic (α, β) -metric

A generalized form of an (α, β) -metric on an n -dimensional manifold M defined as

$$F = \alpha \left(1 + \frac{\beta}{\alpha} \right)^p, \quad (3)$$

is known as the class p-power (α, β) -metrics [19], where $p \neq 0$ is a real constant. In the present work, we consider $p = 3$ in equation (3) and obtain a special class of (α, β) -metric in the form of

$$F = \frac{(\alpha + \beta)^3}{\alpha^2}, \quad (4)$$

and referred as a cubic (α, β) -metric in an n -dimensional manifold M , and an n -dimensional Finsler space F^n equipped with a cubic (α, β) -metric is known as a Finsler space with a cubic (α, β) -metric [18].

6. Exponential (α, β) -metric

In the present investigation, we explore a special type of Finsler exponential (α, β) -metric. We explore the fundamental characteristics of Finsler space associated with this metric.

The exponential (α, β) -metric can be expressed as follows:

$$F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{\frac{-\beta}{\alpha}}. \quad (5)$$

7. (ρ, β, η) -metrics

The present investigation is devoted to the study of a Kropina–Randers change within the framework of (ρ, β, η) -metrics in Finsler geometry, motivated by the rich geometric implications of Randers-type deformations and their capacity to generate new Finsler structures from known ones. The Kropina–Randers change of a Finsler metric F is defined as follows:

$$\bar{F} = F + \eta,$$

where $F = \frac{\rho^2}{\beta}$ is the classical Kropina metric, $\rho = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric arising from the symmetric positive definite tensor $a_{ij}(x)$, and $\beta = b_i(x)y^i$, $\eta = \eta_i(x)y^i$ are two linearly independent differential 1-forms on an n -dimensional smooth manifold M . Consequently, the resulting changed metric assumes the explicit form $\bar{F} = \frac{\rho^2}{\beta} + \eta$, which belongs to the general class of (ρ, β, η) -metrics.

3 Definition of the Problem

Finsler geometry, as a natural generalization of Riemannian geometry, provides a rich and flexible framework for studying geometric structures that go beyond the classical quadratic restriction. A particularly important class of Finsler metrics is the (α, β) -metrics, which depend on a Riemannian metric α and a differential 1-form β . Despite significant progress in the theory of such metrics, several fundamental problems concerning their curvature properties, projective and conformal transformations, and classification remain open and incompletely understood.

Despite the extensive literature on (α, β) -metrics, the geometric behavior of several important special metrics under projective changes are not yet completely understood. For metrics such as the cubic metric, the square root metric, the exponential (α, β) -metric, and the generalized Matsumoto metric, questions regarding projective flatness, vanishing of flag curvature $K = 0$, isotropic S-curvature and classification as Berwald or Douglas spaces have only been partially addressed in the existing literature. A systematic and unified treatment of these problems, covering both classical and newly introduced metrics, is therefore needed.

Beyond the classical (α, β) -framework, there is a growing need to investigate more general metric structures involving two or more differential 1-forms. Metrics depending on a Riemannian metric α together with 1-forms $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$, as well as Kropina–Randers type metrics constructed from $\rho = \sqrt{a_{ij}(x)y^i y^j}$ with 1-forms $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$, present new geometric challenges. In particular, deriving their spray coefficients, Cartan connections, and conditions for projective correspondence and Douglas

type classification require new analytical tools and methods.

Furthermore, the study of $\mathbb{R}$ -complex Finsler spaces, where scalar constraints are restricted to the real field $\lambda \in \mathbb{R}$, introduces additional complexity in understanding the interplay between complex structures and cubic-type deformations. The determination of metric tensors, their inverses, and the conditions for non-Hermitian characteristics in this setting remains a largely unexplored area.

In view of these open problems, the present thesis investigates the differential geometry of Finsler spaces equipped with special metrics. The work aims to establish necessary and sufficient conditions for projective and conformal changes, classify special Finsler spaces through curvature conditions, and extend the geometric framework to multi-form and complex Finsler structures, thereby contributing to a deeper and more complete understanding of modern Finsler geometry.

4 Objective and Scope of work

The main aim and scope of this research involves the study of geometric properties of Finsler spaces equipped with special (α, β) -metrics, with particular emphasis on projective transformations, curvature properties, and the classification of special Finsler spaces such as Berwald spaces, Douglas spaces, weakly Berwald spaces.

- To study the geometric properties of Finsler spaces under projective change and the concept of projectively flat Finsler spaces with the cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$, the square root metric $F = \sqrt{\alpha(\alpha+\beta)}$, and the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\beta}$, and to derive necessary and sufficient conditions for projective correspondence, isotropic S -curvature, and isotropic Berwald curvature.
- To introduce and study a new class of exponential (α, β) -metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ and to determine the necessary and sufficient conditions under which it defines a Berwald space, a Douglas space and a weakly Berwald space, and further to examine its locally dually flat and projectively flat properties in the presence of isotropic S -curvature.
- To investigate the curvature properties of the cubic metric and the generalized Matsumoto metric, specifically to establish conditions for locally projectively flat structures under constant Ricci curvature, vanishing flag curvature $K = 0$, and to prove the conditions under which these metrics are locally Minkowskian.
- To extend the classical (α, β) -framework to a broader class of Finsler metrics depending on a Riemannian metric α together with two differential 1-forms $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$, to derive their spray coefficients and Cartan connection, and to establish conditions for projective correspondence and Douglas-type classification, thereby generating novel Douglas metrics in the setting of multi-form Finsler manifolds.
- To study the S -curvature and Mean Berwald curvature for a class of homogeneous exponential (α, β) -metrics and to derive explicit analytical formulas for the rate of change of distortion along geodesics, thereby providing a complete characterization of the average Berwald curvature.

- To investigate the geometric structure of $\mathbb{R}$ -complex Finsler spaces with cubic-type metrics, including the rigorous derivation of fundamental metric tensors, their determinants and inverses, and to establish conditions for non-Hermitian characteristics, providing a complete analytical profile of the interplay between complex structures and cubic-type metrics.
- To introduce and analyze a new class of Kropina–Randers type Finsler metrics constructed from a Riemannian metric $\rho = \sqrt{a_{ij}(x)y^i y^j}$ together with two differential 1-forms $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$, and to determine the necessary and sufficient conditions under which they are locally projectively flat and of Douglas type.

5 Original contribution by the Thesis

The present thesis makes several original contributions to the theory of Finsler geometry in the areas of projective transformations, curvature theory, and classification of special Finsler spaces. The main contributions are summarized as follows.

Contribution 1: We study the projective change between the cubic metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ to Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\beta}$ and square root metric $F = \sqrt{\alpha(\alpha+\beta)}$ to Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\beta}$. We find the necessary and sufficient conditions for projective transformation and we obtain explicit conditions for isotropic S -curvature and isotropic Berwald curvature.

Contribution 2: We introduce a new exponential (α, β) -metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ and study its geometric structure in detail. We find the necessary and sufficient conditions under which this metric defines a Berwald space, a Douglas space, and a weakly Berwald space.

Contribution 3: We investigate the curvature properties of the cubic metric and the generalized Matsumoto metric under constant Ricci curvature. We obtain original results on locally projectively flat structures and we derive the conditions under which these metrics admit vanishing flag curvature $K = 0$.

Contribution 4: We study the locally dually flat and projectively flat properties of cubic and exponential (α, β) -metrics in the presence of isotropic S -curvature. We prove the necessary and sufficient conditions under which these metrics are locally Minkowskian and we obtain their associated curvature invariants.

Contribution 5: We extend the classical (α, β) -framework to a broader class of metrics involving two differential 1-forms $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$. We derive the spray coefficients and Cartan connection for this class and we find that this framework generates numerous novel Douglas metrics.

Contribution 6: We study the S -curvature and Mean Berwald curvature of a class of homogeneous exponential (α, β) -metrics. We derive an explicit analytical formula for the rate of change of distortion along geodesics and we obtain a complete characterization of the average Berwald curvature.

Contribution 7: We investigate the $\mathbb{R}$ -complex Finsler spaces with cubic-type metrics restricted to the real field $\lambda \in \mathbb{R}$. We derive the fundamental metric tensors, their determinants and inverses, and we find the necessary conditions for non-Hermitian characteristics in this setting.

Contribution 8: We introduce a new class of Kropina–Randers-type metrics constructed from $\rho = \sqrt{a_{ij}(x)y^i y^j}$ with two differential 1-forms $\beta = b_i(x)y^i$ and $\eta = \eta_i(x)y^i$. We find

the necessary and sufficient conditions under which these metrics are locally projectively flat and we obtain their Douglas-type classification.

The results obtained in this thesis have been compiled into research papers published or communicated in peer-reviewed journals, and we believe these findings open new and promising directions for further research in modern Finsler geometry.

6 Methodology of Research and Result discussion

Let $F^n = (M, F(\alpha, \beta))$ represent an n -dimensional Finsler space and $R^n = (M, \alpha)$ represent the corresponding Riemannian space, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$. Let γ_{jk}^i be the Christoffel symbols with respect to α , and the covariant differentiation with respect to γ_{jk}^i be denoted by $(;)$. From the differential 1-form $\beta(x, y) = b_i(x)y^i$, we define

$$2r_{ij} = b_{i;j} + b_{j;i}, \quad 2s_{ij} = b_{i;j} - b_{j;i} = (\partial_j b_i - \partial_i b_j), \\ s_j^i = a^{ir} s_{rj}, \quad b^i = a^{ir} b_r, \quad b^2 = a^{rs} b_r b_s.$$

6.1 Introduction

The introductory chapter provides the background and significance of the study. It also highlights the objectives and presents the overall framework of the thesis.

6.2 On a Special Finsler (α, β) -metric

To obtain the conditions for projective change and projectively flat Finsler spaces with (α, β) -metrics, we have employed the following results:

Projective transformation between Cubic (α, β) -metric and Kropina-metric

Proposition 2 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be a cubic (α, β) -metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have same Douglas tensor if and only if both are Douglas metrics.

Theorem 3 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero colinear 1-forms. Then F is projectively related to $\tilde{F}$ if and only if the following relations hold:

$$(i) \quad G_\alpha^i = \tilde{G}_\alpha^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{\tau \alpha^2 ((1 + 6b^2)\alpha^2 - 8\beta^2)b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2\alpha^2},$$

$$(ii) \quad b_{i|j} = \frac{\tau}{3} \{ (1 + 6b^2)a_{ij} - 8b_i b_j \},$$

$$(iii) \quad \tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i],$$

Corollary 4 Consider two (α, β) -metrics $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ on an n -dimension manifold M with $n > 2$. Then F is projectively related to $\tilde{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ have the following relation

$$G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{\tau \alpha^2 ((1 + 6b^2)\alpha^2 - 8\beta^2) b^i}{\alpha^2 - \alpha\beta - 8\beta^2 + 6b^2 \alpha^2}, \quad (6)$$

where $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M .

Projective change with curvature properties of Cubic (α, β) -metric

Theorem 5 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S -curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:

- (i) $G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
- (ii) β is parallel with respect to α , that is, $b_{i|j} = 0$;
- (iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 6 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Let us Suppose that the Finsler metric F possesses isotropic Berwald curvature. Then F is projectively related to $\tilde{F}$ if and only if the following requirements are satisfied:

1. $G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
2. β is parallel with respect to α , that is, $b_{i|j} = 0$;
3. $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 7 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S -curvature.

Theorem 8 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and F has isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S -curvature.

Projective transformation between Square root-metric and Kropina-metric

Proposition 9 Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a square root (α, β) -metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. The Finsler metrics F and $\tilde{F}$ have same Douglas tensor if and only if both are Douglas metrics.

Theorem 10 Let $F = \sqrt{\alpha(\alpha + \beta)}$ be a square root-metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be a Kropina metric on a manifold M with dimension $n \geq 3$, where α and $\tilde{\alpha}$ are two Riemannian metrics, β and $\tilde{\beta}$ are two non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if the following relations hold:

$$(i) \quad G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{2\tau \alpha^2 ((3\beta^2 - (b^2 - 4)\alpha^2) b^i)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2},$$

$$(ii) \quad b_{i|j} = 2\tau \{ (b^2 - 4)a_{ij} - 3b_i b_j \},$$

$$(iii) \quad \tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i],$$

Corollary 11 Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Then F is projectively related to $\tilde{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\tilde{\alpha}$ have the following relation

$$G_{\alpha}^i = \tilde{G}_{\tilde{\alpha}}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}_i + r_{00} \tilde{b}^i] + \theta y^i - \frac{2\tau \alpha^2 ((3\beta^2 - (b^2 - 4)\alpha^2) b^i)}{6\beta^2 + 12\alpha\beta + 8\alpha^2 - 2b^2\alpha^2}, \quad (7)$$

where $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M .

Projective change with curvature properties of Square root metric

Theorem 12 Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic S -curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:

$$(i) \quad G_{\alpha}^i = \tilde{G}_{\alpha}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i;$$

(ii) β is parallel with respect to α , that is, $b_{i|j} = 0$;

(iii) $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 13 Let $F = \sqrt{\alpha(\alpha + \beta)}$ and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ be two (α, β) -metrics on a manifold M with dimension $n > 2$, where α and $\tilde{\alpha}$ are two Riemannian metrics, $\beta = b_i y^i$ and $\tilde{\beta}$ are two non-zero 1-forms. Suppose that F has isotropic Berwald curvature. Then F is projectively related to $\tilde{F}$ if and only if the following conditions hold:

1. $G_{\alpha}^i = \tilde{G}_{\alpha}^i + \frac{1}{2\tilde{b}^2} [\tilde{\alpha}^2 \tilde{s}^i + \tilde{r}_{00} \tilde{b}^i] + \theta y^i$;
2. β is parallel with respect to α , that is, $b_{i|j} = 0$;
3. $\tilde{s}_{ij} = \frac{1}{\tilde{b}^2} [\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i]$, where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α and $\tilde{b}^i := \tilde{a}^{ij} \tilde{b}_j$, $\tilde{b} := \|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\theta = \theta_i y^i$ is a 1-form on M .

Theorem 14 Let $F = \sqrt{\alpha(\alpha + \beta)}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and $\tilde{F}$ has isotropic Berwald curvature. Then F has isotropic Berwald curvature if and only if F has isotropic S -curvature.

Theorem 15 Let $F = \sqrt{\alpha(\alpha + \beta)}$ be projectively equivalent to $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$ and F has isotropic Berwald curvature. Then $\tilde{F}$ has isotropic Berwald curvature if and only if $\tilde{F}$ has isotropic S -curvature.

6.3 Finser space with (α, β) -metric

We find the conditions under which the Finsler space characterized by the exponential-metric

$$F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}} \quad (8)$$

defines a Berwald space, a Douglas space, and a weakly Berwald space.

The condition to be a Berwald space

Theorem 16 A Finsler space with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Berwald space if and only if $b_{i;j} = 0$.

The condition to be a Douglas space

Theorem 17 An n -dimensional Finsler space F^n with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space, if and only if

1. $\alpha^2 \not\equiv 0 \pmod{\beta} : b_{j;i} = 0$.
2. $\alpha^2 \equiv 0 \pmod{\beta} : n = 2$ and $b_{j;i} = 0$, where $\alpha^2 = \beta \delta$, $\delta = d_i(x)y^i$ and $k = k(x)$.

Theorem 18 If an n -dimensional Finsler space F^n with an exponential (α, β) -metric $F(\alpha, \beta) = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ is a Douglas space, then F^n is a Berwald space.

The Condition to be a Weakly-Berwald space

Theorem 19 A Finsler space endowed with the metric (8) is weakly Berwald space if and only if the following are satisfied;

- (1) $\alpha^2 \not\equiv 0 \pmod{\beta}$, which yields $r_{ij} = 0$ and $s_j = 0$.
- (2) $\alpha^2 \equiv 0 \pmod{\beta}$ indicates that $n=2$, $b^2 = 0$, and $r_{ij} = 0, s_j = 0$ are pleased, where $\alpha^2 = \beta \delta, \delta = d_i y^i$.

6.4 Projectively Flat of Finsler-metric with constant flag curvature

The objective of this research is to derive the conditions under which generalized Matsumoto and cubic (α, β) -metrics are projectively flat with zero flag curvature.

Lemma 20 For a Projectively flat cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$, D^i is given by,

$$D^i = \frac{\{(r_{00}P_2 + s_0P_3)y^i + (r_{00}P_4 + s_0P_5)B^i + s^iP_6\}}{P_1}. \quad (9)$$

Theorem 21 The S -curvature of Projectively flat cubic (α, β) -metric is given by,

$$S = (n+1) \left\{ \frac{\{(r_{00}P_2 + s_0P_3) + (r_{i0}P'_4 + k_{i00}P'_5)B^i + K^i_{i0}P'_6\}}{P_1} - dp(y) \right\}, \quad (10)$$

where $p := \ln \sqrt{1 - \|\beta\|^2}$.

Projectively Flat Cubic (α, β) -metric with $K = 0$

Theorem 22 Let $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ be a locally projectively flat cubic (α, β) -metric, suppose that F has constant Ricci curvature $\text{Ric} = (n-1)\lambda F^2$, then

- (a) F has flag curvature $K = \lambda = 0$,
- (b) α has curvature $\bar{K} = \mu = 0$.

Theorem 23 If $F = \frac{(\alpha + \beta)^3}{\alpha^2}$ is locally projectively flat cubic (α, β) -metric, then F is locally Minkowskian.

Geometric Equations for the Projectively Flat Generalized Matsumoto Metric

Lemma 24 For a projectively generalized Matsumoto metric $F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$, D^i is given by,

$$D^i = \frac{(A_1 r_{00} + A_2 s_0) y^i + (B_2 r_{00} + B_3 s_0) B^i + B_4 s^i}{B_5}. \quad (11)$$

Theorem 25 The S-curvature of projectively flat generalized Matsumoto metric is provided by,

$$S = (n+1) \left\{ \frac{(A_1 r_{00} + A_2 s_0) + (B'_2 r_{i0} + B'_3 k_{i00}) B^i + B'_4 k_{i0}^i}{B_5} - dp(y) \right\}, \quad (12)$$

where $p := \ln \sqrt{1 - \|\beta\|^2}$.

Projectively Flat Generalized Matsumoto Metric with $K = 0$

Theorem 26 Let $F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$ be a locally projectively flat generalized Matsumoto metric.

Assume that F has constant Ricci curvature $Ric = (n-1)\lambda F^2$, then

(a) F has Flag curvature $K = \lambda = 0$,

(b) α has curvature $\bar{K} = \mu = 0$.

Theorem 27 If $F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m}$ is locally Projectively flat generalized Matsumoto metric, then F is locally Minkowskian.

6.5 Locally dually flatness properties in Finsler (α, β) -Metrics

In this present section, we obtained the criteria for a non-Riemannian cubic (α, β) -metric and exponential-metric to be locally dually flat and projectively flat, and we characterized these conditions using isotropic S-curvature.

Characterization of Locally Dually Flat Cubic (α, β) - Metric

Theorem 28 The cubic (α, β) -metric on an n -dimensional manifold M ($n \geq 3$) is locally dually flat if and only if the following conditions are came together.

1. $s_{l0} = \frac{1}{3}(\beta \theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l) \right] \alpha^2 + \frac{1}{3}(13 + 90b^2)\tau\beta^2$;
3. $G_\alpha^l = \frac{1}{3}[2\theta + 43\tau\beta]y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 - 15\tau\beta^2 b^l$.

Theorem 29 Let F be a cubic (α, β) -metric of non-Randers type defined on an n -dimensional manifold M , where $n \geq 3$. Assume further that F has an isotropic S -curvature of the form $S = (n+1)cF$, where $c = c(x)$ is a scalar function that depends only on position, and that F is locally dually flat. When expressed in an adapted coordinate system, it follows that F is locally projectively flat under these circumstances. Furthermore, the spray coefficients satisfy $G^i = 0$.

Theorem 30 Consider a non-Riemannian cubic (α, β) -metric F on an n -dimensional manifold M , where $n \geq 3$. If F is locally dually flat and admits isotropic S -curvature, the condition $S = (n+1)cF$, where $c = c(x)$ is a scalar function, holds only if F is locally Minkowskian.

Characterization of Locally Dually Flat Exponential-Metric

Theorem 31 The indispensable and adequate conditions for an exponential (α, β) -metric to be locally dually flat on a n -dimensional manifold ($n \geq 3$) are stated as follows:

1. $s_{l0} = \frac{1}{3}(\beta\theta_l - \theta b_l)$;
2. $r_{00} = \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{12}(-56 - 159b^2)\tau\beta^2$;
3. $G_\alpha^l = \frac{1}{3}(2\theta + 7\tau\beta)y^l + \frac{1}{3}(\theta^l\tau b^l)\alpha^2 + \frac{53}{8}\tau\beta^2 b^l$.

Theorem 32 If F non-Randers type exponential (α, β) -metric on n -dimensional manifold M ($n \geq 3$) and locally dually flat with isotropic S curvatures $S = (n+1)cF$, then F is a locally projectively flat in adapted coordinate system and $G^i = 0$.

Theorem 33 If F non-Riemannian exponential (α, β) -metric on n -dimensional manifold M ($n \geq 3$) and locally dually flat with isotropic S curvature then $S = (n+1)cF$ iff the structure is locally Minkowskian.

6.6 A new class of Finsler metrics

This Section demonstrates how to create a new type of Finsler metrics that aren't always (α, β) -metrics. The metrics mentioned above are determined by a Riemannian metric and two 1-forms, $\beta = b_i(x)y^i$ and $\gamma = d_i(x)y^i$. We also receive a variety of results when employing this form of Finsler metric.

Theorem 34 An (F, β) -metric of the form $\tilde{F} = (\alpha + \gamma)\phi(s)$, is projectively flat if and only if the following condition is satisfied:

$$\left\{ \begin{array}{l} 2 \left\{ \frac{1+2s-\rho}{(1+s)^2} \right\} \left\{ \left(a_{ir} + \frac{2Y_r d_i \alpha^2 - Y_i Y_r \gamma}{\alpha^3} + \frac{Y_i d_r}{\alpha} + d_i d_r \right) \right. \\ \left. - \left(\frac{Y_i}{\alpha} + d_i \right) \left(\frac{Y_r}{\alpha} + d_r \right) \right\} G^r - \frac{2(\alpha + \gamma)s_{i0}}{(1+s)^2} + \\ \left. \frac{2(b_i - s(\frac{Y_i}{\alpha} + d_i))((1+2s-\rho)r_{00} + 2(\alpha + \gamma)s_0)}{(1+s)^2((1+2s-\rho)(1+s) + 2(b^2 - s^2))} = 0. \end{array} \right. \quad (13)$$

Corollary 35 Let $F = \sqrt[m]{A}$ with $(m > 3)$ be an m -th root Finsler metric defined on an open subset $U \subset \mathbb{R}^n$, where A is an irreducible function. Consider the Matsumoto type (F, β) -metric given by

$$\tilde{F} = F \left(\frac{1}{1 + \frac{\beta}{F}} \right)$$

with $\beta = b_i(x)y^i$. Then $\tilde{F}$ is locally projectively flat if and only if the following condition holds:

$$\begin{cases} \frac{-m(m-1)}{(1+s)^5} [((1+2s-\rho)(1+s) + 2(b^2 - s^2)) (1+2s)] \\ y_i A_0 A^{1-\frac{4}{m}} + \frac{m}{(1+s)^5} [((1+2s-\rho)(1+s) + 2(b^2 - s^2)) (1+2s)] \\ (A_{0i} - A_{xi}) A^{1-\frac{2}{m}} - \frac{2}{(1+s)^5} \left[((1+2s-\rho)(1+s) + 2(b^2 - s^2)) s_{i0} \right. \\ \left. - 2s_0 \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right] A^{\frac{1}{m}} + \frac{2(1+2s)(b_i - s(\frac{Y_i}{\alpha} + d_i))r_{00}}{(1+s)^5} = 0. \end{cases}$$

Theorem 36 Let (M, F) be a Douglas space. A Matsumoto type (F, β) -metric $\tilde{F} = (\alpha + \gamma)\phi \left(\frac{\beta}{F} \right)$, where b_i is the h -vector, is Douglas if and only if the following condition holds:

$$\begin{cases} H^{ij} = \left(\frac{-(\alpha+\gamma)}{1+2s-\rho} \right) \left(s_0^i y^j - s_0^j y^i \right) + \left[\frac{(1+2s-\rho)r_{00} + 2(\alpha+\gamma)s_0}{\lambda(1+s)^3(1+2s-\rho)} \right] \\ \left\{ b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right\} (b^i y^j - b^j y^i). \end{cases} \quad (14)$$

Matsumoto type (F, β) -metrics

Lemma 37 For Finsler metric $F = \alpha + \gamma$ and 1-form $\beta = b_i y^i$ with h -vector b_i on manifold M with $\|\beta\|_F < b_0$. $\tilde{F} = F\phi(s)$ is a Finsler metric if and only if the positive C^∞ function $\phi(s) = \frac{1}{1+s}$ satisfies

$$\frac{1+2s-\rho}{(1+s)^2} > 0, \quad \frac{(1+2s-\rho)(1+s) + 2(b^2 - s^2)}{(1+s)^3} > 0, \quad (15)$$

when $n \geq 3$ or $\frac{(1+2s-\rho)(1+s) + 2(b^2 - s^2)}{(1+s)^3} > 0$, when $n = 2$, where $s = \frac{\beta}{F}$ and b are arbitrary numbers with $|s| \leq b < b_0$.

Corollary 38 Let M be an n -dimensional manifold. For Finsler metric $F = \alpha + \gamma$ and 1-form $\beta = b_i y^i$ with $\|\beta\|_F < b_0$. $\tilde{F} = (\alpha + \gamma) \left(\frac{1}{1+s} \right)$ is a Matsumoto type Finsler metric if and only if the positive C^∞ function satisfies $\frac{1}{1+s} + \frac{s}{(1+s)^2} > 0$. $\frac{1}{1+s} + \frac{s}{(1+s)^2} + \frac{2(b^2 - s^2)}{(1+s)^3} > 0$; when $n \geq 3$, or $\frac{1}{1+s} + \frac{s}{(1+s)^2} + \frac{2(b^2 - s^2)}{(1+s)^3} > 0$, when $n = 2$, s and b are arbitrary number.

Spray coefficients of Matsumoto type (F, β) -metrics

Lemma 39 The system of algebraic equations

$$(i) \quad \left\{ \left(\frac{1+2s-\rho}{(1+s)^2} \right) \left(\frac{a_{ir}\alpha^2 - a_{ir}}{\alpha^3} \right) + \frac{2}{(1+s)^3(\alpha+\gamma)} \left(b_i - s \left(\frac{Y_i}{\alpha} + d_i \right) \right) \right. \\ \left. (b_r - s \left(\frac{Y_r}{\alpha} + d_r \right)) \right\} A^r = B_i$$

$$(ii) \left\{ \frac{1}{1+s} \left(\frac{Y_r}{\alpha} + d_r \right) - \frac{1}{(1+s)^2} \left(b_r - s \left(\frac{Y_r}{\alpha} + d_r \right) \right) \right\} A^r = B,$$

has a single approach A^r for given B and B_i such that $B_i y^i = 0$. The approach is provided by

$$A^i = \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} B^i + (1+s) \left(B + \frac{(\alpha+\gamma)B_r b^r}{\lambda(1+s)^2} \right) l^i + \frac{-2(\alpha+\gamma)B_r b^r m^i}{(1+s)(1+2s-\rho)\lambda},$$

where $B^i = g^{il} B_l$, $l^i = g^{il} l_i$ and $m^i = g^{il} m_l$.

Theorem 40 Let $\tilde{F}$ be a Matsumoto type (F, β) -metric with h -vector b_i . Then the spray coefficients of $\tilde{F}$ are given by

$$\left\{ \begin{aligned} 2\tilde{G}^i &= 2G^i - \frac{2(\alpha+\gamma)}{1+2s-\rho} s_0^i + \frac{2b^i \{ (1+2s-\rho)r_{00} + 2(\alpha+\gamma)s_0 \}}{(1+s)^3(1+2s-\rho)\lambda} \\ &+ \left\{ \frac{-(1+4s-\rho)((1+2s-\rho)r_{00} + 2(\alpha+\gamma)s_0)}{(1+s)^3(1+2s-\rho)\lambda} \right\} l^i \end{aligned} \right\} \quad (16)$$

Corollary 41 Let $\tilde{F}$ be a Matsumoto type (F, β) -metric. Then the spray coefficients of $\tilde{F}$ are given by

$$\left\{ \begin{aligned} 2\tilde{G}^i &= 2G^i - \frac{2(\alpha+\gamma)}{1+2s} s_0^i + \left\{ \frac{(-1-4s)((1+2s)r_{00} + 2(\alpha+\gamma)s_0)}{(1+3s+2b^2)(1+2s)} \right\} l^i \\ &+ \frac{2b^i \{ (1+2s)r_{00} + 2(\alpha+\gamma)s_0 \}}{(1+3s+2b^2)(1+2s)} \end{aligned} \right\} \quad (17)$$

Cartan connection of Matsumoto type (F, β) -metrics

$$\left\{ \begin{aligned} \bar{C}_{jk}^i &= C_{jk}^i + \frac{\eta'_0}{2\eta_F} h_{jk}^i + \frac{\eta'_0}{2\eta_F} m^i m_j m_k - \frac{1}{2\eta_F} \left\{ (2\rho\eta + \eta'(b^2 - s^2)) h_{jk} + \right. \\ &\left. (2\eta' + \eta'_0(b^2 - s^2)) m_j m_k \right\} \left\{ \left(\frac{\bar{\delta}}{1+\bar{\delta}b^2} + \frac{\bar{\mu}\bar{\lambda}^2}{1+\bar{\mu}\bar{Y}^2} \right) b^i + \frac{\bar{\mu}\bar{\lambda}^2}{1+\bar{\mu}\bar{Y}^2} l^i \right\}. \end{aligned} \right\} \quad (18)$$

$$\left\{ \begin{aligned} \bar{N}_j^i &= N_j^i + \frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} Q_j^i + (s+1) l^i \left(Q_j + \frac{(\alpha+\gamma)Q_{rj} b^r}{\lambda(1+s)^2} \right) \\ &- \frac{2(\alpha+\gamma)(Q_{rj} b^r m^i)}{\lambda(1+s)(1+2s-\rho)}. \end{aligned} \right\} \quad (19)$$

$$\left\{ \begin{aligned} \bar{\Gamma}_{jk}^i &= \Gamma_{jk}^i + \left(\frac{(\alpha+\gamma)(1+s)^2}{1+2s-\rho} \right) M_{jk}^i + (s+1) \left(M_{jk} + \frac{(\alpha+\gamma)M_{rjk} b^r}{\lambda(1+s)^2} \right) l^i \\ &- \frac{2(\alpha+\gamma)M_{rjk} b^r m^i}{\lambda(1+2s-\rho)(1+s)}. \end{aligned} \right\} \quad (20)$$

Theorem 42 Let $C\bar{\Gamma} = (\bar{\Gamma}_{jk}^i, \bar{N}_j^i, \bar{C}_{jk}^i)$ be the Cartan connection for the Finsler space $(M, \tilde{F})$. Hence, the Cartan connection is entirely defined by the equations (18), (19) and (20).

Lemma 43 [12] Let $F = \sqrt[m]{A}$ ($m > 2, m \neq 5$), be an m -th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the equation $\Psi A^{1-\frac{4}{m}} + \Omega A^{1-\frac{2}{m}} + \Theta A^{\frac{1}{m}} = 0$ holds, where Ψ, Ω and Θ are homogeneous polynomials in y . Then $\Psi = \Omega = \Theta = 0$.

6.7 Curvatures on homogeneous Finsler space with Finsler metrics

Theorem 44 Consider a reductive Homogeneous Finsler Space represented as G/H , where the Lie algebra decomposes as $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. The G -invariant exponential (α, β) -metric is expressed as $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$. The S -curvature is determined by this metric.

$$S(H, y) = \frac{\xi_1}{\xi_2} \left(\frac{\langle [v, y]_m, y \rangle}{\alpha} + \frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \langle [v, y]_m, v \rangle \right). \quad (21)$$

Corollary 45 Let $(G/H, F)$ represent a homogeneous generalized Matsumoto-Metric under conditions similar to those stated in Theorem 44. A homogeneous generalized Matsumoto-Metric exhibits isotropic S -curvature if and only if the S -curvature of F is equal to zero.

Mean Berwald Curvature

Theorem 46 Let $(G/H, F)$ be a reductive homogeneous Finsler Space with decomposition of Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and $F = \alpha e^{\frac{\beta}{\alpha}} + \beta e^{-\frac{\beta}{\alpha}}$ be a G -invariant exponential (α, β) -metric. Then Mean Berwald Curvature is given by

$$\left\{ \begin{aligned} E_{i,j}(H, y) &= \frac{1}{2} \left[\frac{\partial}{\partial y^i} \left(\frac{1}{\alpha} \frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2} \frac{y_j}{\alpha} \right) \langle [v, y]_m, y \rangle + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) \right. \\ &= \left(\frac{1}{\alpha} \Omega_{ij} - \frac{y_i}{\alpha^3} \Omega_j - \frac{y_j}{\alpha^3} \Omega_i - \frac{\Omega}{\alpha^3} \delta_i^j + \frac{3\Omega}{\alpha^5} y_i y_j \right) \langle [v, y]_m, y \rangle + \left(\frac{1}{\alpha} \Omega_j - \frac{\Omega y_j}{\alpha^3} \right) \\ &\quad (\langle [v, v_i]_m, y \rangle + \langle [v, y]_m, v_i \rangle) + \left(\frac{1}{\alpha} \Omega_i - \frac{\Omega y_i}{\alpha^3} \right) \\ &\quad (\langle [v, v_j]_m, y \rangle + \langle [v, y]_m, v_j \rangle) + \frac{\Omega}{\alpha} (\langle [v, v_j]_m, v_i \rangle + \langle [v, v_i]_m, v_j \rangle) \\ &\quad + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_{ij} + \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \Omega_j u_{y^i} \right. \\ &\quad + \Omega_i \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{y^j} + \Omega u_{y^i y^j} \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \\ &\quad \left. + \Omega u_{y^j} u_{y^i} \frac{A_1}{A_2} \right\} \langle [v, y]_m, v \rangle + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_i + \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) \right. \\ &\quad \left. u_{y^i} \Omega \right\} \langle [v, v_j]_m, v \rangle + \left\{ \left(\frac{e^u + e^{-u} - ue^{-u}}{e^u - ue^u + u^2 e^{-u}} \right) \Omega_j + \Omega \left(\frac{e^{2u} + 2u - 2 - 2ue^{-2u} + u^2 e^{-2u}}{(e^u - ue^u + u^2 e^{-u})^2} \right) u_{y^j} \right\} \\ &\quad \left. \langle [v, v_i]_m, v \rangle \right]. \end{aligned} \right. \quad (22)$$

6.8 On R-complex Finsler space with Special Finsler metrics

Theorem 47 The invariants of R -complex Finsler space with cubic (α, β) -metric: $\rho_0, \rho_1, \rho_{-2}, \rho_{-1}$ and μ_0 are given by:

$$\rho_0 = \frac{1}{2} \alpha^{-1} L_\alpha = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6),$$

$$\begin{aligned}\rho_1 &= \frac{1}{2}L_\beta = \frac{3(\alpha + \beta)^5}{\alpha^4}, \\ \rho_{-2} &= \frac{\alpha L_{\alpha\alpha} - L_\alpha}{4\alpha^3} = \frac{-27\alpha(\alpha + \beta)^5 + 15\alpha^2(\alpha + \beta)^4 + 12(\alpha + \beta)^6}{2\alpha^8}, \\ \rho_{-1} &= \frac{L_{\alpha\beta}}{4\alpha} = \frac{30\alpha(\alpha + \beta)^4 - 24(\alpha + \beta)^5}{4\alpha^6}, \\ \mu_0 &= \frac{L_{\beta\beta}}{4} = \frac{30(\alpha + \beta)^4}{4\alpha^4}.\end{aligned}$$

Subscripts 0, 1, -2, -1 and 0 gives us the degree of homogeneity of these invariants.

Fundamental tensor of R-Complex Finsler space with Cubic (α, β) -metric

Theorem 48 For a non-Hermitian R-Complex Finsler space with cubic (α, β) -metric $F = \frac{(\alpha + \beta)^3}{\alpha^2}$, following identities holds good:

1. The contravariant tensor $\tilde{g}^{ij}$ of the fundamental tensor $\tilde{g}_{ij}$ is:

$$\tilde{g}^{ji} = \frac{2\alpha^6}{(6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6)} a^{ji} + A_1 \eta^i \eta^j + A_2 (b^i \eta^j + b^j \eta^i) + A_3 b^i b^j + \frac{A_4}{A_5},$$

$$\begin{aligned}2. \det(a_{ij} + \frac{-3\alpha\beta + 12\beta^2}{\alpha^2(2\alpha^2 - 2\alpha\beta - 4\beta^2)} l_{il} j) &+ \frac{15\alpha^2}{2\alpha^2 - 2\alpha\beta - 4\beta^2} b_i b_j + \frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2} \eta_i \eta_j) \\ &= [1 + (\gamma(1 + \gamma k_1 + \varepsilon k_2) + \varepsilon(\frac{r_7}{r_8} + \frac{r_9}{r_{10}})) \sqrt{\frac{3\alpha - 12\beta}{2\alpha^2 - 2\alpha\beta - 4\beta^2}}] \\ &\times [1 + \omega + \frac{(-3\alpha\beta + 12\beta^2)\varepsilon^2}{2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + (-3\alpha\beta + 12\beta^2)\gamma}] \times (1 + 2\alpha^4 - 2\alpha^3\beta - 4\alpha^2\beta^2 + \\ &(-3\alpha\beta + 12\beta^2)\gamma) \det(a_{ij}).\end{aligned}$$

Proposition 49 In a non-Hermitian R-complex Finsler space with cubic (α, β) -metric following properties holds good:

$$\gamma + \bar{\gamma} = l_i \eta^i + l_{\bar{j}} \eta^{\bar{j}} = a_{ij} \eta^j \eta^i + a_{\bar{j}\bar{k}} \eta^{\bar{k}} \eta^{\bar{j}} = 2\alpha^2, \quad (23)$$

$$\varepsilon + \bar{\varepsilon} = b_j \eta^j + b_{\bar{j}} \eta^{\bar{j}} = 2\beta, \quad \delta = \varepsilon, \quad (24)$$

where,

$$\begin{aligned}l_i &= a_{ij} \eta^j, \quad \eta_i = \frac{1}{2\alpha^6} (6\alpha(\alpha + \beta)^5 - 4(\alpha + \beta)^6) a_{ij} \eta^i + \frac{3(\alpha + \beta)^5}{\alpha^4} b_i, \\ \gamma &= a_{jk} \eta^j \eta^k = l_k \eta^k, \quad \varepsilon = b_j \eta^j, \quad b^k = a^{jk} b_j, \quad b_l = b^k a_{kl}, \quad \delta = a_{jk} \eta^j b^k = l_k b^k, \quad l_j = a^{jl} l_i = \eta^j.\end{aligned}$$

6.9 On (ρ, β, η) -metrics with Kropina – Rander changes

Proposition 50 The fundamental tensor for an (ρ, β, η) -metric for a Kropina-Randers Change can be also represented as follows:

$$\left\{ \begin{aligned} g_{ij} &= \sigma \left\{ a_{ij} + \delta_1 b_i b_j + \delta_2 \eta_i \eta_j + \delta_0 (b_i + \eta_i)(b_j + \eta_j) + \frac{\sigma_2}{\sigma} (\rho_i + \frac{\sigma_1}{\sigma_2} b_i + \frac{\bar{\sigma}_1}{\sigma_2} \eta_i) \right. \\ &\quad \left. (\rho_j + \frac{\sigma_1}{\sigma_2} b_j + \frac{\bar{\sigma}_1}{\sigma_2} \eta_j) \right\}. \end{aligned} \right.$$

Where,

$$\sigma = \left(\frac{1}{u} + \bar{u}\right)\left(\frac{2}{u}\right), \quad \sigma_1 = \frac{-4}{u^3} - \frac{2\bar{u}}{u^2}, \quad \sigma_2 = \frac{4}{u^2}, \quad \bar{\sigma}_1 = \frac{2}{u}, \quad \delta_0 = \frac{u^2}{2(1+u\bar{u})}\left(\frac{-1}{u^2} - \frac{-2(-4-2u\bar{u})}{4u^2}\right),$$

$$\delta_1 = \frac{u^2}{2(1+u\bar{u})}\left(\frac{3u+2\bar{u}}{u^4} - \frac{(-4-2u\bar{u})^2}{4u^4}\right) - \frac{u^2}{2(1+u\bar{u})}\left(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2}\right), \quad \delta_2 = \frac{-u^2}{2(1+u\bar{u})}\left(\frac{-1}{u^2} - \frac{2(-4-2u\bar{u})}{4u^2}\right).$$

Proposition 51 *Let M symbolize an n -dimensional manifold. A Finsler metric F is defined as $F = \rho\Theta(s, \bar{u})$, where $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$, with $u = \frac{\beta}{\rho}$ and $\bar{u} = \frac{\eta}{\rho}$. This metric applies to Riemannian ρ and 1-forms $\beta = b_i y^i$ and $\eta = \eta_i y^i$ under the conditions that $\|\beta\|_\rho < c_1$ and $\|\eta\|_\rho < c_2$. The condition $\theta - u\bar{u} \geq 0$ is only valid if the corresponding positive C^∞ function $\Theta = \Theta(u, \bar{u})$ is fulfilled.*

$$\Theta := \frac{2}{u} > 0, \quad \eta > 0. \quad (25)$$

Spray coefficients of F

Lemma 52 *The algebraic equation system (i) $l_i A^r = B_i$ and (ii) $l_r A^r = B$ has a distinct solution for A^r when B and B_i are defined, with the condition that $B_i y^i = 0$. The solution may be stated as*

$$A^i = (\rho_r A^r) \rho^i + \frac{\rho}{\Theta} B^i - \frac{\rho}{\Theta \Gamma} (\mu_1 h^i + \mu_2 \bar{h}^i), \quad (26)$$

where,

$$B^i = a^{il} B_l, \quad h^i = a^{il} h_l, \quad \bar{h}^i = a^{il} \bar{h}_l \quad \text{and}$$

$$\Theta := \frac{2}{u},$$

$$\mu_1 := \left(\frac{2}{u^3}\right) B_r b^r,$$

$$\mu_2 := 0.$$

Proposition 53 *The spray coefficients $\mathcal{G}^i$ are connected to ${}^\rho \mathcal{G}^i$*

$$\mathcal{G}^i = {}^\rho \mathcal{G}^i + \frac{\rho}{A} \left[\left(\frac{-2}{u^3 \bar{u}} \right) \beta_{s_0^i} - \left(\frac{1}{u^2 \bar{u}^2} \right) \eta_{s_0^i} \right] + \frac{1}{2\Gamma} [\Gamma_1 b^i + \Gamma_2 \eta^i + \frac{1}{\Theta} \Gamma_3 \rho^i], \quad (27)$$

where,

$$\begin{cases} \Gamma = \frac{2}{u} + (b^2 - u^2) \frac{2}{u^3}, \\ \Gamma_1 = \left[\frac{2}{u^3} \right] R^\beta, \\ \Gamma_2 = 0, \\ \Gamma_3 = \left\{ \frac{-4}{u^3} - \frac{2\bar{u}}{u^2} + \left(\frac{-2}{u^3} \right) (\theta - u\bar{u}) \right\} R^\beta + \left\{ \frac{2}{u} + \frac{2}{u^3} (b^2 - u^2) \right\} R^\eta. \end{cases} \quad (28)$$

and

$$R^\beta := \beta_{r_{00}} - (\rho u) \left\{ \left(\frac{-1}{u^2} \right) \beta_{s_0} + \eta_{s_0} \right\};$$

$$R^\eta := \eta_{r_{00}} - (\rho u) \left\{ \left(\frac{-1}{u^2} \right) \beta_{s_0} + \eta_{s_0} \right\}.$$

Projectively flat (ρ, β, η) -metrics

Example 54 We assume a (ρ, β, η) -metric in the following form: $F = \frac{\rho^2}{\beta} + \eta$, $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$. Let $c_1 > 0$ and $l_1 > 0$ be the largest numbers such that

$$\Theta = \frac{2}{u}, \quad \Gamma = \frac{2b^2}{u^3} > 0. \quad |u| < b < c_1, \quad |\bar{u}| < g < l_1. \quad (29)$$

Lemma 55 An (ρ, β, η) -metric $F = \rho\Theta(u, \bar{u})$, where $\Theta(u, \bar{u}) = \frac{1}{u} + \bar{u}$, $u = \frac{\beta}{\rho}$ and $\bar{u} = \frac{\eta}{\rho}$, is projectively flat on an open subset $u \subseteq R^n$ if and only if

$$\rho h_{ij} \rho \mathcal{G}^i + \frac{\beta}{2} \left[\frac{-\rho^2}{\beta^2} \beta_{s_{j0}} + \eta_{s_{j0}} \right] + \frac{2\rho^6}{4b^2\beta^7} [\beta\beta_{r_{00}} + \rho^2\beta_{s_0} - \beta^2\eta_{\bar{u}}] h_j = 0. \quad (30)$$

Lemma 56 The (ρ, β, η) -metric $F = \frac{\rho^2}{\beta} + \eta$ is considered to be locally projectively flat if and only if β is parallel with regard to ρ and η is closed.

Proposition 57 Consider F , the metric introduced in Example 54. We can prove (ρ, β, η) -metric $F = \frac{\rho^2}{\beta} + \eta$ is Douglas if and only if β is parallel with respect to ρ and η is closed.

6.10 Conclusion

The present thesis has made significant contributions to the differential geometry of Finsler spaces equipped with special metrics. The major conclusions drawn from this research are summarized as follows:

- The projective transformation between the cubic (α, β) -metric $F = \frac{(\alpha+\beta)^3}{\alpha^2}$ and the Kropina metric $\tilde{F} = \frac{\tilde{\alpha}^2}{\beta}$, as well as between the square root metric $F = \sqrt{\alpha(\alpha+\beta)}$ and the Kropina metric, was established along with their associated S -curvature properties.
- The exponential (α, β) -metric $F = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ was analyzed and conditions for it to be a Berwald space, a Douglas space, and a weakly Berwald space were determined.
- Locally projectively flat cubic and generalized Matsumoto (α, β) -metrics with constant Ricci curvature were classified, and the existence of such metrics with vanishing flag curvature ($K = 0$) was established.
- Conditions under which the cubic (α, β) -metric and the exponential (α, β) -metric are locally dually flat and projectively flat were derived using isotropic S -curvature. Explicit formulas for S -curvature and Mean Berwald curvature for the exponential (α, β) -metric were also obtained.
- A novel class of Finsler metrics extending the traditional (α, β) -framework was introduced by defining $\tilde{F} = \frac{F^2}{F+\beta}$, where $F = \alpha + \gamma$ is a Randers metric. Spray coefficients and Cartan connection were derived, and necessary and sufficient conditions for projective correspondence were established, yielding a significant family of new Douglas metrics.

- For $\mathbb{R}$ -complex Finsler spaces with cubic-type metrics, the fundamental metric tensors $\tilde{g}_{ij}$ and $\tilde{g}_{i\bar{j}}$, their determinants, and inverses were completely derived. Conditions for non-Hermitian characteristics were also identified.
- A new Kropina–Randers metric $F = \frac{\rho^2}{\beta} + \eta$ involving two 1-forms β and η was introduced. Spray coefficients were derived and conditions for local projective flatness were established, and the metric was characterized as a Douglas space.

7 Achievements with respect to objectives

The achievements with respect to objectives are summarized as follows:

- Necessary and sufficient conditions for projective transformations between cubic, square root, and Kropina metrics were derived, including conditions for isotropic S -curvature and isotropic Berwald curvature.
- A new exponential (α, β) -metric $F = \alpha e^{\beta/\alpha} + \beta e^{-\beta/\alpha}$ was introduced and conditions for a Berwald space, a Douglas space, and a weakly Berwald space were established.
- Curvature properties of cubic and generalized Matsumoto metrics were studied under constant Ricci curvature and conditions for vanishing flag curvature ($K = 0$) were obtained.
- Conditions for locally dually flat and projectively flat cubic and exponential (α, β) -metrics were derived, including criteria for locally Minkowskian classification.
- Finsler geometry was extended to a broader class of two 1-form metrics involving β and γ . Spray coefficients and Cartan connections were derived, and novel Douglas metrics were obtained.
- Explicit formulas for S -curvature and Mean Berwald curvature for homogeneous exponential (α, β) -metrics were derived using the algebraic structure of the underlying Lie algebra.
- Metric tensors, their determinants, and inverses for $\mathbb{R}$ -complex Finsler spaces with cubic-type metrics were analyzed and conditions for non-Hermitian characteristics were identified.
- A new Kropina–Randers-type metric constructed from two 1-forms was introduced and conditions for locally projectively flat and Douglas-type classifications were established.

8 List of publications arising from the Thesis

1. B. K. Tripathi, D. Patel and T. N. Pandey, *On a Weakly-Berwald Space with Exponential Form of (α, β) -Metric*, Journal of Rajasthan Academy of Physical Sciences, **22** (2023), 97-113.
(Web of Science/UGC-Care list). <https://raops.org.in/epapers/june239.pdf>

2. B. K. Tripathi, D. Patel and V. K. Chaubey, *Projectively Flat Cubic (α, β) -Metric with Constant Flag Curvature $K=0$* , Surveys in Mathematics and its Applications, **19** (2024), 167-178. (SCOPUS/UGC-Care list). <https://www.utgjiu.ro/math/sma/v19/p1910.pdf>
3. B. K. Tripathi and D. Patel, *Berwald and Douglas Spaces of a Finsler Space With an Exponential form of (α, β) - metric*, Korean Journal of Mathematics, **32** (2024), 661-671. (Web of science/Scopus). <https://kkms.org/index.php/kjm/article/view/1531>.
4. B. K. Tripathi, V. K. Chaubey and D. Patel, *On R-Complex Finsler Space with Cubic (α, β) -metric*, Journal of Dynamical Systems and Geometric Theories, **22** (2024), 47-63. (Web of Science). <https://doi.org/10.47974/JDSGT-2024-003>
5. B. K. Tripathi, V. K. Chaubey and D. Patel, *Locally dually flatness properties in Cubic (α, β) -metric*, Mathematical Foundations of Computing, **8**(4) (2025) , 507-517. (Scopus, Web of Science).10.3934/mfc.2024027
6. D. Patel and B. K. Tripathi, *Projective change between Cubic (α, β) -metric and Kropina metric*, Bulletin of Mathematical Analysis and Applications, **17** (1) (2025), 1-16. (Scopus, Web of Science). <http://www.bmathaa.org/repository/docs/BMAA17-1-1.pdf>.
7. D. Patel and B. K. Tripathi, *Projective change between Square root metric and Kropina metric*, Bulletin of the Institute of Mathematics Academia Sinica , **21** (2026), 1-19. (Web of Science). 10.21915/BIMAS.2026101.

Paper presented in Conferences:

- Presented a paper entitled ***Projectively Flat Cubic (α, β) -Metric with Constant Flag Curvature $K = 0$*** in the International Conference On Mathematical Modelling, Simulation and Nonlinear Dynamics - 2024, Department of Mathematics, Bharathiar University, Coimbatore, Tamilnadu, India.
- Presented a paper entitled ***On R-Complex Finsler Space with Cubic (α, β) -metric*** in the International Geometry Symposium, Munzur University, Turkey.
- Presented a paper entitled ***On R-Complex Finsler Space with Matsumoto (α, β) -metric*** in the Department of Mathematics North-Eastern Hill University, Shilong, Meghalaya, India.
- Presented a paper entitled ***Projectively Flat Generalized Matsumoto Metric with $K = 0$*** in the International Symposium on Differential Geometry and its Application, Munzur University, Turkey.

- Presented a paper entitled *Projective Change between Cubic (α, β) -Metric and Kropina Metric* in the International Symposium in Memory of the 100th Anniversary of Gazi University, Turkey.

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